NUMBER OF EIGENVALUES FOR DISSIPATIVE SCHRÖDINGER OPERATORS UNDER PERTURBATION

XUE PING WANG

Abstract. In this article, we prove for a class of dissipative Schrödinger operators $H = -\Delta + V(x)$ with a complex-valued potential $V(x)$ on $\mathbb{R}^n$, $n \geq 2$, and $\Im V(x) \leq 0$ and $\Im V \neq 0$ that 0 is not an accumulating point of the eigenvalues of H. If $\Im V$ is sufficiently small, we show that $N(V) = N(\Re V) + k$, where k is the multiplicity of the zero resonance of the selfadjoint Schrödinger operator $-\Delta + \Re V$ and $N(W)$ the number of eigenvalues of $-\Delta + W$, counted according to their algebraic multiplicity.

Résumé. Dans cet article, nous démontrons que zéro n'est pas point d'accumulation des valeurs propres pour une classe d'opérateurs de Schrödinger dissipatifs $H = -\Delta + V(x)$ sur $\mathbb{R}^n$, $n \geq 2$, avec un potentiel complexe $V(x)$ tel que sa partie imaginaire vérifie : $\Im V(x) \leq 0$ et $\Im V \neq 0$. Si $\Im V$ est suffisamment petit, nous montrons que $N(V) = N(\Re V) + k$, où k est la multiplicité de la résonance au seuil zéro de l'opérateur de Schrödinger autoadjoint $-\Delta + \Re V$ et $N(W)$ le nombre des valeurs propres de $-\Delta + W$, comptées selon leur multiplicité algébrique.

1. Introduction

Consider the Schrödinger operator $H = -\Delta + V(x)$ with a complex-valued potential $V(x) = V_1(x) - iV_2(x)$ on $L^2(\mathbb{R}^n)$, $n \geq 2$, where V_1 and V_2 are real measurable functions. V and H are called dissipative if $V_2(x) \geq 0$ and $V_2(x) > 0$ on some non trivial open set. Assume that V is a $-\Delta$-compact perturbation. H is then closed with domain $D(H) = D(-\Delta)$. Let $\sigma(H)$ (resp., $\sigma_{ess}(H)$, $\sigma_d(H)$) denote the spectrum (resp., essential spectrum, discrete spectrum) of H. By Weyl's essential spectrum theorem, one has $\sigma_{ess}(H) = [0, \infty[$ and the spectrum of H is discrete in $\mathbb{C} \setminus [0, \infty[$, consisting of eigenvalues with finite multiplicity which may accumulate to any point of $[0, \infty[$. For real-valued potentials V, it is well-known that if $V(x)$ decays like $O(|x|^{-\rho})$ for some $\rho > 2$, the eigenvalues of $H_1 = -\Delta + V_1(x)$ can not accumulate to 0 (cf. [9]). In this work, we prove that this is still true for dissipative Schrödinger operators when $n \geq 3$. We also study the number of eigenvalues of H when $\Im V$ is regarded as a small perturbation. Throughout this work, eigenvalues are counted according to their algebraic multiplicity.

The minimal assumptions used in this work are as follows. Suppose that $n \geq 2$, V_1 and V_2 are real functions satisfying the estimates

$$|V_j(x)| \leq C\langle x\rangle^{-\rho_j}, \quad V_2(x) \geq 0 \text{ and } V_2 \neq 0, \tag{1.1}$$

2000 *Mathematics Subject Classification.* 35J10, 35P15, 47A55.

Key words and phrases. Complex eigenvalues, threshold resonance, non-selfadjoint operators, dissipative Schrödinger operators.

Research supported in part by the French national research project NONAa, No. ANR-08-BLAN-0228-01 and by Qian Ren project at Nanjing University.

for some $\rho_j > 1$, $j = 1, 2$. Here $\langle x \rangle = (1 + |x|^2)^{1/2}$; the real part of the potential is allowed to have critical decay:

$$V_1(x) = \frac{q(\theta)}{r^2} + O(\langle x \rangle^{-\rho_1'}), \quad |x| > R, \tag{1.2}$$

for some $R > 0$ and $\rho_1' > 2$, where $r = |x|$, $x = r\theta$, $\theta \in \mathbb{S}^{n-1}$ and $q(\theta)$ is a real continuous function on $\mathbb{S}^{n-1}$ such that the lowest eigenvalue, μ_1, of $-\Delta_{\mathbb{S}^{n-1}} + q(\theta)$ on $\mathbb{S}^{n-1}$ verifies

$$\mu_1 > -\frac{(n-2)^2}{4}. \tag{1.3}$$

Set $\nu_1 = \sqrt{\mu_1 + \frac{(n-2)^2}{4}}$. Note that if $n \geq 3$ and V_1 satisfies (1.1) for some $\rho_1 > 2$, (1.2) and (1.3) are satisfied with $q = 0$ and $\mu_1 = 0$. For $n = 2$, the condition (1.3) requires the potential to be positive in some sense when $|x|$ is large enough. Rapidly decaying potentials are excluded when $n = 2$.

For potentials satisfying (1.1), (1.2) and (1.3), we say that zero is a resonance of H if the equation $Hu = 0$ has a solution $u \in H^{1,-s} \setminus L^2$ for any $s > 1$ ($H^{1,-s}$ is the weighted first order Sobolev space with the weight $\langle x \rangle^{-s}$) and u is then called a resonant state. As for selfadjoint operators (cf. [4]), zero is called a regular point of H if it is neither an eigenvalue nor a resonance of H (notice however that dissipative Schrödinger operators H have no real eigenvalues). For the selfadjoint operator H_1 with a critically decaying potential V_1, zero resonance may appear in any space dimension $n \geq 2$ with arbitrary multiplicity depending on $q(\theta)$ (see [14, 15]). The following result says that this can not happen for dissipative Schrödinger operators.

Theorem 1.1. *Let $n \geq 2$. Under the conditions (1.1)-(1.3) with $\rho_1' > 2$ and $\rho_2 > 2$, zero is a regular point of H. The eigenvalues of H can not accumulate to zero and there exists $c_0 > 0$ such that the limits*

$$R(\lambda \pm i0) = \lim_{\epsilon \to 0_+} R(\lambda \pm i\epsilon) \tag{1.4}$$

exist in $\mathcal{L}(-1, s; 1, -s)$, $s > 1$, uniformly in $\lambda \in [-c_0, c_0]$.

,

For the notation $\mathcal{L}(-1, s; 1, -s)$, see the end of Introduction. In [8], A. Laptev and O. Safronov deduce from their estimates on complex eigenvalues that if $n = 3$ and $V_2 \geq 0$ is integrable, the eigenvalues of $-\Delta - iV_2$ can not accumulate to zero. The limiting absorption principle of Schrödinger operators with complex-valued potentials is studied in [6, 11] at $\lambda > 0$ and outside some exceptional set of measure zero in $]0, \infty[$ (see also [12]). Recently, the limiting absorption principle from the upper half-complex plane for each $\lambda > 0$ is proved in [10] for abstract dissipative operators without such an implicit condition.

The next result of this work is on the number of eigenvalues of a dissipative Schrödinger operator when the imaginary part of the potential is small. Denote $H(\gamma) = H_1 - i\gamma V_2$ where $\gamma > 0$ is a small parameter. Let $N(\gamma)$ (resp. N_1) be the total number of the complex eigenvalues of $H(\gamma)$ (resp., H_1). It is easy to show that under the same conditions as in Theorem 1.1, if 0 is a regular point of H_1, then

$$N(\gamma) = N_1 \tag{1.5}$$

for $0 < \gamma \le \gamma_0$. See Proposition 3.1. A more interesting question is the case when zero happens to be an eigenvalue or a resonance of H_1. For the class of potentials V_1 under consideration, zero resonance of H_1 may appear in any space dimension with arbitrary multiplicity. The interaction between resonant states makes the threshold spectral analysis rather difficult. See [15] for the resolvent expansion. In this work, we only study a particular case where

$$\left\{\nu = \sqrt{\mu + \frac{(n-2)^2}{4}}, \mu \in \sigma(-\Delta_{\mathbb{S}^{n-1}} + q)\right\} \cap]0,1] = \{\nu_1\} \tag{1.6}$$

with $\nu_1 = \sqrt{\mu_1 + \frac{(n-2)^2}{4}}$. The condition (1.6) is satisfied if $q(\theta) = q_0$ is an appropriate constant and it ensures that if zero is a resonance of H_1, then it is simple. Let φ_0 be a normalized eigenfunction (which can be taken to be positive) of $-\Delta_{\mathbb{S}^{n-1}} + q$ associated with μ_1. Set

$$W_1(x) = V_1(x) - \frac{q(\theta)}{r^2}, \quad \eta_0(x) = \frac{\varphi_0(\theta)}{r^{\frac{n-2}{2}-\nu_1}}, \quad x = r\theta.$$

Theorem 1.2. *Assume (1.1) - (1.3) with $\rho_1' > 4$ and $\rho_2 > 4$ and (1.6).*

(a). Assume that zero is an eigenvalue but not a resonance of H_1. Then

$$N(\gamma) = N_1 \tag{1.7}$$

for $0 < \gamma < \gamma_0$. Here N_1 is the total number of eigenvalues of H_1, including the zero eigenvalue.

(b). Assume that zero is a resonance but not an eigenvalue of H_1 and that

$$\nu_1 \in [\frac{1}{2}, 1] \quad \text{and} \quad \overline{\langle W_1\eta_0, \phi\rangle}\langle V_2\eta_0, \phi\rangle < 0. \tag{1.8}$$

Then there exits $\gamma_0 > 0$ such that

$$N(\gamma) = N_1 + 1, \tag{1.9}$$

for $0 < \gamma < \gamma_0$. Here N_1 is the total number of negative eigenvalues of H_1.

Note that $\langle W_1\eta_0, \phi\rangle \neq 0$ if ϕ is a resonant state ([14, 15]) and the condition

$$\overline{\langle W_1\eta_0, \phi\rangle}\langle V_2\eta_0, \phi\rangle < 0$$

is independent of the choice of ϕ. If $n = 3$ or 4 and if the condition (1.1) is satisfied with $\rho_1 > 2$, one has $q = 0$, $\nu_1 = \frac{1}{2}$ or 1, respectively, $W_1 = V_1$ and η_0 is constant: $\eta_0 = \frac{1}{\sqrt{|\mathbb{S}^{n-1}|}}$. The condition (1.8) is then simplified as

$$\overline{\langle V_1, \phi\rangle}\langle V_2, \phi\rangle < 0. \tag{1.10}$$

In particular, if $V_2 = -V_1$, one has

$$\overline{\langle V_1, \phi\rangle}\langle V_2, \phi\rangle = -|\langle V_1, \phi\rangle|^2 < 0$$

for any resonant state ϕ, because $\langle V_1, \phi\rangle \neq 0$ by the characterization of resonant states. As a consequence of Theorem 1.2, we deduce that under the conditions that $n = 3, 4,$

$V_1 = -V_2$ verifying the condition (1.1) with $\rho_1 = \rho_2 > 4$ and zero is an eigenvalue or a resonance of H_1, the number of eigenvalues of $H(\gamma) = -\Delta + (1 + i\gamma)V_1$ is given by

$$N(\gamma) = \begin{cases} N_1, & \text{if zero is not a resonance of } H_1; \\ N_1 + 1, & \text{if zero is a resonance of } H_1. \end{cases} \tag{1.11}$$

for $\gamma > 0$ sufficiently small. An example of the potential V_1 for which zero is a resonance but not an eigenvalue of H_1 is given in Section 3.

Theorem 1.1 is proved in Section 2. As a consequence, we deduce a global resolvent estimate on the whole real axis which may be useful to study the long-time quantum dynamics of the semigroup. The number of eigenvalues under dissipative perturbation is studied in Section 3. A Breit-Wigner type resolvent estimate is given near the eigenvalues. The main attention is payed to the case where zero eigenvalue and zero resonance of the selfadjoint operator H_1 are present. The techniques used in the proof of the both theorems are threshold spectral analysis.

Notation. $H^{r,s}$, $r, s \in \mathbb{R}$, denotes the weighted Sobolev space of order r defined by $H^{r,s} = \{f \in \mathcal{S}'(\mathbb{R}^n); \langle x\rangle^s (1-\Delta)^{r/2} f \in L^2\}$ equipped with the natural norm noted as $\|\cdot\|_{r,s}$. The dual product between $H^{r,s}$ and $H^{-r,-s}$ is identified with L^2-scalar product. Denote $H^{0,s} = L^{2,s}$ and $H^{r,0} = H^r$. $\mathcal{L}(r, s; r', s')$ is the space of continuous linear operators from $H^{r,s}$ to $H^{r',s'}$ and $\mathcal{L}(r, s) = \mathcal{L}(r, s; r, s)$.

2. Spectral properties near the threshold

The following result is essential to prove Theorem 1.1.

Lemma 2.1. *Let $s \in [0, 1[$. Suppose that the condition (1.1) is satisfied with $\rho_j > s+1$, $j = 1, 2$. Then $u \in H^{1,-s}$ and $Hu = 0$ imply $u = 0$.*

Proof. Let $\rho' = \min\{\rho_1, \rho_2\}$. Then $\rho' - s > 1$ and one has $-\Delta u = -Vu \in L^{2,\rho'-s}$, $H_1 u = iV_2 u \in L^{2,\rho'-s}$. The equation $Hu = 0$ gives

$$\langle u, H_1 u\rangle = i\langle u, V_2 u\rangle. \tag{2.1}$$

We want to show that $\langle H_1 u, u\rangle$ is a real number, although u is not in the domain of the selfadjoint operator H_1. To do this, we need to show that $\nabla u \in L^2$. Notice first that since $u \in L^{2,-s}$, $\Delta u \in L^{2,s} \subset L^{2,-s}$ and since $\langle x\rangle^{-s}\nabla(1-\Delta)^{-1}\langle x\rangle^s$ is bounded on L^2, one has

$$\langle x\rangle^{-s}\nabla u = (\langle x\rangle^{-s}\nabla(1-\Delta)^{-1}\langle x\rangle^s)\langle x\rangle^{-s}(1-\Delta)u \in L^2.$$

Therefore $e^{-\epsilon\langle x\rangle}\nabla u \in L^2$ for any $\epsilon > 0$ and one has

$$\|e^{-\epsilon\langle x\rangle}\nabla u\|^2 = \langle e^{-2\epsilon\langle x\rangle}u, -\Delta u\rangle + 2\epsilon\langle e^{-2\epsilon\langle x\rangle}u, \frac{x}{\langle x\rangle}\cdot\nabla u\rangle. \tag{2.2}$$

Since $u \in L^{2,-s}$ for some $s < 1$, one has

$$|\epsilon\langle e^{-2\epsilon\langle x\rangle}u, \frac{x}{\langle x\rangle}\cdot\nabla u\rangle| \le M\epsilon^{1-s}\|\langle x\rangle^{-s}u\|\|e^{-\epsilon\langle x\rangle}\nabla u\|, \tag{2.3}$$

with $M = \sup_{r\geq 0} r^s e^{-r}$. It follows that for $\epsilon_0 > 0$ small enough

$$\|e^{-\epsilon\langle x\rangle}\nabla u\|^2 \leq \frac{1}{1-M\epsilon^{1-s}}|\langle e^{-2\epsilon\langle x\rangle}u, -\Delta u\rangle| + \frac{M\epsilon^{1-s}}{1-M\epsilon^{1-s}}\|\langle x\rangle^{-s}u\|^2, \quad 0<\epsilon\leq\epsilon_0. \tag{2.4}$$

Since $-\Delta u \in L^{2,s}$ and $u \in L^{2,-s}$,

$$|\langle e^{-2\epsilon\langle x\rangle}u, -\Delta u\rangle| \leq \|u\|_{L^{2,-s}}\|\Delta u\|_{L^{2,s}} \quad \forall \epsilon > 0.$$

We deduce that $\sup_{0<\epsilon\leq\epsilon_0}\|e^{-\epsilon\langle x\rangle}\nabla u\|^2 < \infty$ which implies $\nabla u \in L^2$ and

$$\|\nabla u\|^2 \leq |\langle u, -\Delta u\rangle|.$$

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(x) = 1$ for $|x| \leq 1$, 0 for $|x| \geq 2$. let $\chi_R(x) = \chi(x/R)$, $R > 1$. Then $\langle \chi_R u, -\Delta(\chi_R u)\rangle \geq 0$ and

$$\begin{aligned}\langle \chi_R u, -\Delta(\chi_R u)\rangle &= \langle \chi_R u, -\chi_R\Delta u\rangle + \langle \chi_R u, [-\Delta, \chi_R]u\rangle \\ &\to \langle u, -\Delta u\rangle, \quad R\to\infty.\end{aligned}$$

When taking the limit, we used $u \in L^{2,-s}$, $\Delta u \in L^{2,s}$ with $s < 1$ and $\nabla u \in L^2$. This proves that $\langle u, -\Delta u\rangle \geq 0$. In particular, $\langle u, H_1 u\rangle = \langle u, -\Delta u\rangle + \langle u, V_1 u\rangle$ is a real number.

It follows from (2.1) that $\langle H_1 u, u\rangle = 0$ and $\langle V_2 u, u\rangle = 0$. Since $V_2 \geq 0$ and $V_2 \neq 0$, one has $V_2 u = 0$ and $u(x) = 0$ for x in a non trivial open set Ω. Now u is solution to the equation $H_1 u = 0$. We can apply the unique continuation theorem (see [3]) to H_1 to conclude $u = 0$ on $\mathbb{R}^n$. □

The same argument as that used in Lemma 2.1 shows that H has no real eigenvalues.

Corollary 2.2. *Under the conditions of Theorem 1.1, zero is a regular point of H: if $Hu = 0$ and $u \in H^{1,-s}$ for any $s > 1$, then $u = 0$.*

Proof. For $u \in H^{1,-s}$, $\forall s > 1$, and $Hu = 0$, one can expand u in terms of the eigenfunctions of $-\Delta_{\mathbb{S}^{n-1}} + q(\theta)$ to show that

$$u(x) = \frac{\psi(\theta)}{r^{\frac{n-2}{2}+\nu_1}} + v, \qquad \text{for } r = |x| \text{ large},$$

with $\nu_1 = \sqrt{\mu_1 + \frac{(n-2)^2}{4}} > 0$, $\psi \in L^2(\mathbb{S}^{n-1})$ and $v \in L^2(|x| > 1)$. See Theorem 4.1 in [14]. This means that u is in fact in $H^{1,-s}$ for any $s \in]1-\nu_1, 1[$. Since $\min\{\rho_1, \rho_2\} = 2 > 1+s$, Lemma 2.1 can be applied to conclude that $u = 0$. □

For $z_0 \in \mathbb{C}$ and $r > 0$, denote $D(z_0, r) = \{z \in \mathbb{C}; |z - z_0| < r\}$, $D_\pm(z_0, r) = D(z_0, r) \cap \mathbb{C}_\pm$ and $D'(0, r) = D(0, r) \setminus [0, r[$.

Proof of Theorem 1.1. Let $\chi_1(x)^2 + \chi_2(x)^2 = 1$ be a partition of unity on $\mathbb{R}^n$ with $\chi_1 \in C_0^\infty$ such that $0 \leq \chi_j \leq 1$ and $\chi_1(x) = 1$ for $|x| \leq 1$. (1.3) implies that the form defined by $-\Delta + \frac{q(\theta)}{r^2}$ on $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ is positive. Let H_0 denote its Friedrich's extension. Then the following generalized Hardy's inequality holds: there exists $c > 0$ such that

$$\langle \frac{1}{r^2}u, u\rangle \leq c\langle u, H_0 u\rangle, \quad \forall\, u \in D(H_0). \tag{2.5}$$

Set

$$\tilde{R}_0(z) = \chi_1(-\Delta + 1 - z)^{-1}\chi_1 + \chi_2(H_0 - z)^{-1}\chi_2,$$

and

$$K(z) = \tilde{R}_0(z)(H - z) - 1,$$

for $z \in D'(0,\delta)$. One has

$$\begin{aligned} K(z) &= \chi_1(-\Delta + 1 - z)^{-1}([\Delta, \chi_1] + (V-1)\chi_1) \\ &\quad + \chi_2(H_0 - z)^{-1}([\Delta, \chi_2] + (V - \frac{q(\theta)}{r^2})\chi_2). \end{aligned}$$

Note that $(-\Delta + 1 - z)^{-1}$ is holomorphic for z near 0 and the asymptotics of $(H_0 - z)^{-1}$ are computed in [14] (see Theorem A.1 of Appendix A). In particular, (1.3) ensures that the $\ln z$-term is absent. One deduces that the limit

$$F_0 = \lim_{z \to 0, z \notin \mathbb{R}_+} \tilde{R}_0(z)$$

exists in $\mathcal{L}(-1, s; 1, -s)$ for any $s > 1$. Since $\rho_1' > 2$ and $\rho_2 > 2$, $K(z)$ is a compact operator-valued function on $H^{1,-s}$, $1 < s < \min\{\rho_1'/2, \rho_2/2\}$, holomorphic in $D'(0,\delta)$, continuous up to the boundary and

$$K(z) = K_0 + O(|z|^{\delta_0})$$

in $\mathcal{L}(1, -s; 1, -s)$ for some $\delta_0 > 0$, where $K_0 = \lim_{z \to 0, z \notin \mathbb{R}_+} K(z)$ is a compact operator. By the same method, one can show that the limits $\widetilde{R}_0(\lambda \pm i0)$ and $K(\lambda \pm i0)$ exist uniformly for λ near 0.

We claim that -1 is not an eigenvalue of K_0. In fact, let $u \in H^{1,-s}$ for any $s > 1$ such that $K_0 u = -u$. A standard ellipticity argument shows that $u \in H^2_{\text{loc}}$. By the expression of K_0, one sees that

$$Hu = H_0(u + K_0 u) = 0$$

in the open set where $\chi_2(x) = 1$. Therefore $w := Hu$ is of compact support and

$$F_0 w = (1 + K_0)u = 0.$$

In particular,

$$\langle w, F_0 w \rangle = q_1(w) + q_2(w) = 0, \tag{2.6}$$

where

$$\begin{aligned} q_1(w) &= \lim_{\lambda \to 0_-} \langle w_1, (-\Delta + 1 - \lambda)^{-1} w_1 \rangle \quad \text{and} \\ q_2(w) &= \lim_{\lambda \to 0_-} \langle w_2, (-\Delta + \frac{q(\theta)}{r^2} - \lambda)^{-1} w_2 \rangle \end{aligned}$$

with $w_j = \chi_j w$, $j = 1, 2$. For each $\lambda < 0$, one has $\langle w_1, (-\Delta + 1 - \lambda)^{-1} w_1 \rangle \geq 0$ and $\langle w_2, (-\Delta + \frac{q(\theta)}{r^2} - \lambda)^{-1} w_2 \rangle \geq 0$ (by (2.5)). Taking the limit $\lambda \to 0$, one obtains $q_1(w) \geq 0$ and $q_2(w) \geq 0$. (2.6) gives then $q_j(w) = 0$, $j = 1, 2$. Since $w \in L^2$, $q_1(w) = 0$ gives that $\chi_1 w = 0$. This proves $w = Hu = 0$. By Corollary 2.2, one has $u = 0$. Therefore -1 is not an eigenvalue of K_0 and $(1 + K_0)^{-1}$ exists in $\mathcal{L}(1, -s; 1, -s)$,

$s \in]1, \min\{\rho_1'/2, \rho_2/2\}[$. An argument of perturbation shows that $1 + K(z)$ is invertible for $z \in D'(0,\delta)$ with $\delta > 0$ small enough and

$$\sup_{z \in \in D'(0,\delta)} \|(1 + K(z))^{-1}\|_{\mathcal{L}(1,-s;1,-s)} < \infty. \tag{2.7}$$

It follows from the equation $\widetilde{R}_0(z)(H - z) = 1 + K(z)$ that H has no eigenvalues in $D'(0,\delta)$ and one has

$$R(z) = (1 + K(z))^{-1}\widetilde{R}_0(z), \quad z \in D'(0,\delta), \tag{2.8}$$

and for any $s > 1$, $\sup_{z \in D'(0,\delta)} \|\langle x\rangle^{-s} R(z)\langle x\rangle^{-s}\| < \infty$. In addition, The existence of the limits

$$(1 + K(\lambda \pm i0))^{-1} \;\; = \;\; \lim_{\epsilon \to 0_+} (1 + K(\lambda \pm i\epsilon))^{-1}$$

in $\mathcal{L}(1,-s;1,-s)$, $s \in]1, \min\{\rho_1'/2, \rho_2/2\}[$ uniformly for $\lambda \in \mathbb{R}$ and λ near 0 implies the existence of the limits $R(\lambda \pm i0)$ in $\mathcal{L}(-1,s;1,-s)$, $s > 1$, for λ in a neighborhood of 0. □

Remark that $R(\lambda + i0) = R(\lambda - i0)$ for $\lambda < 0$ since H has no spectrum in $]-\infty, 0[$. The same is true for $\lambda = 0$ by the proof given above.

Under the condition (1.1) with $\rho_j > 1$, the limiting absorption principle from the upper-half complex plan for λ away from 0 can be deduced by an argument of perturbation ([1]) or by Mourre's theory for dissipative operators (see [10]) which also applies to dissipative Schrödinger operators with long-range potentials). One deduces from Theorem 1.1 the following

Corollary 2.3. *Under the conditions of Theorem 1.1, one has for any $s > 1$*

$$\|\langle x\rangle^{-s} R(\lambda + i0)\langle x\rangle^{-s}\| \le C_s \langle\lambda\rangle^{-1/2}, \quad \forall\, \lambda \in \mathbb{R}. \tag{2.9}$$

The global resolvent estimate (2.9) can be used to study quantum dynamics of the semigroup e^{-itH}, $t > 0$, such as the rate of time-decay and Kato's smoothness estimate for the semigroup. These problems of scattering nature will be treated in a separate publication.

3. Number of eigenvalues under dissipative perturbation

We are mainly interested in the perturbation of zero eigenvalue and zero resonance of H_1. Let us begin with the easy case where zero is a regular point of the selfadjoint operator H_1 (see also [6]).

Under the conditions (1.1)-(1.3) with $\rho_2 > 2$, $H_1 = -\Delta_1 + V_1$ has only a finite number of eigenvalues:

$$\sigma_1 < \sigma_2 < \cdots < \sigma_k \le 0. \tag{3.1}$$

The resolvent $R_1(z)$ of H_1 verifies for any $\delta > 0$, $s > 1/2$,

$$\|\langle x\rangle^{-s} R_1(z)\langle x\rangle^{-s}\| \le C_\delta |z|^{-\frac{1}{2}}, \tag{3.2}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}_+$ with $|z| > \delta$ and $|z - \sigma_j| > \delta$. If zero is a regular point of H_1, then $\sigma_k < 0$ and one has for any $s > 1$,

$$\|\langle x\rangle^{-s} R_1(z) \langle x\rangle^{-s}\| \le C, \tag{3.3}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}_+$ with $|z| \le \delta$ for some $\delta > 0$ small enough. Since $\rho_2 > 2$, we have

$$\||V_2|^{1/2} R_1(z) |V_2|^{1/2}\| \le C_\delta \tag{3.4}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}_+$ with $|z - \sigma_j| > \delta$, $j = 1, \cdots, k$, if zero is a regular point of H_1.

Proposition 3.1. *Under the conditions (1.1)-(1.3), let $H_1 = -\Delta + V_1$ and $H(\gamma) = H_1 - i\gamma V_2$, $\gamma > 0$. Let $N(\gamma)$ (resp., N_1) denote the number of eigenvalues of $H(\gamma)$ (resp., H_1). Assume that zero is a regular point of H_1. Then there exists some $\gamma_0 > 0$ such that*

$$N(\gamma) = N_1 \tag{3.5}$$

for $0 < \gamma < \gamma_0$. More precisely, for each eigenvalue σ_j of H_1 with multiplicity m_j, there exists $\delta > 0$ such that $H(\gamma)$ has m_j eigenvalues in the disc $\{z; |z - \sigma_j| < \delta\}$ given by

$$z_k = \sigma_j - i\gamma a_k + O(\gamma^2), \quad \gamma \in]0, \gamma_0], \; k = 1, \cdots, m_j, \tag{3.6}$$

for some $a_k > 0$. One has the following Breit-Wigner type resolvent estimate

$$\|R(\mu, \gamma)\| \le \max_{\sigma_j \in \sigma_d(H_1)} \frac{C_\delta}{|\mu - (\sigma_j - i\gamma)|}, \tag{3.7}$$

for $\mu \in] - \infty, -\delta], \delta > 0$ and $\gamma \in]0, \gamma_0]$.

Proof. For each negative eigenvalue $\sigma_j < 0$ of H_1 with multiplicity m_j, a standard perturbation argument can be used to show that $\exists \delta_0 > 0$ such that $H(\gamma)$ has m_j eigenvalues in $D_-(\sigma_j, \delta_0)$ if $\gamma > 0$ is small enough. In fact, let Π_j denote the spectral projection of H_1 associated with the eigenvalue σ_j. Then

$$E(z, \gamma) := ((1 - \Pi_j) H(\gamma) (1 - \Pi_j) - z)^{-1} (1 - \Pi_j)$$

is well defined and is uniformly bounded for $|z - \sigma_j|$ and γ sufficiently small. One has the following Feshbach-Grushin formula:

$$R(z, \gamma) = E(z, \gamma) - (1 + i\gamma E(z, \gamma) V_2) \Pi_j (E_{-+}(z, \gamma))^{-1} \Pi_j (1 + i\gamma V_2 E(z, \gamma)) \tag{3.8}$$

where

$$E_{-+}(z, \gamma) = \Pi_j (z - \sigma_j + i\gamma V_2 - \gamma^2 V_2 E(z, \gamma) V_2) \Pi_j \tag{3.9}$$

See also (3.19) below. The eigenvalues of $H(\gamma)$ in a small disk around σ_j coincide with the zeros of $\det E_{-+}(z, \gamma)$ there. Notice that $\Pi_j V_2 \Pi_j \ge 0$, since $V_2 \ge 0$. This operator is positive definite on Ran Π_j, because if for some $\psi \in$ Ran Π_j, $\Pi_j V_2 \psi = 0$, then $V_2 \psi = 0$, which means $\psi(x) = 0$ for x in some nontrivial open set. But ψ is an eigenfunction of H_1 with eigenvalue σ_j. The unique continuation theorem shows that $\psi = 0$. Consequently, the eigenvalues, $a_1, \cdots, a_{m_j}$, of $\Pi_j V_2 \Pi_j$ on Ran Π_j are strictly positive. One can calculate that $H(\gamma)$ has m_j eigenvalues near σ_j given by

$$z_k = \sigma_j - i\gamma a_k + O(\gamma^2).$$

Since $E_{-+}(z,\gamma)$ can be diagonalized up to $O(\gamma^2)$, we obtain

$$\|(E_{-+}(\mu,\gamma))^{-1}\Pi_j\| \le C\frac{1}{|\mu-(\sigma_j-i\gamma)|} \tag{3.10}$$

uniformly in μ real, $|\mu-\sigma_j|$ and $\gamma>0$ small enough. It follows from (3.8) that (3.7) holds for $R(\mu,\gamma)$ for μ near $\sigma_d(H_1)$. For any $\delta>0$, one has

$$\|R_1(\mu)\| \le C_\delta/|\mu|, \quad \mu\le -\delta \text{ and } \operatorname{dist}(\mu,\sigma_d(H_1))\ge\delta.$$

The global bound (3.7) for $\mu<-\delta$ follows by an argument of perturbation.

For any fixed $\delta_0>0$, let $\Omega=\mathbb{C}_-\setminus(\cup_j D_-(\sigma_j,\delta_0))$. It is easy to see that $H(\gamma)$ has no eigenvalues in Ω when γ is sufficiently small. In fact, by (3.4), there exists $\gamma_0>0$ is such that

$$\gamma_0\||V_2|^{1/2}R_1(z)|V_2|^{1/2}\|<1, \tag{3.11}$$

for all $z\in\Omega$. Then, $H(\gamma)$ has no eigenvalues in Ω if $|\gamma|<\gamma_0$, because if u is an eigenfunction of $H(\gamma)$ associated with the eigenvalue $z_0\in\Omega$,

$$H(\gamma)u=z_0u,$$

then $v=|V_2|^{1/2}u\neq 0$. v is a non zero solution of the equation

$$v=-i\gamma|V_2|^{1/2}R_1(z_0)V_2^{1/2}v, \quad V_2^{1/2}= \text{ sign } V_2|V_2|^{1/2}.$$

This is impossible, because $\|\gamma|V_2|^{1/2}R_1(z_0)|V_2|^{1/2}\|<1$. This proves that the total number of complex eigenvalues of $H(\gamma)$ is equal to the number of negative eigenvalues of H_1. □

As is clear from the proof, the sign restriction on V_2 and γ is not necessary for (3.5), but it is necessary for the Breit-Wigner type resolvent estimate (3.7).

Let us study now the perturbation of zero eigenvalue and zero resonance. Assume that zero is an eigenvalue of $H_1=-\Delta+V_1$ with multiplicity k_0 and a resonance with multiplicity k. k_0 or k may eventually be equal to 0. We want to show $H(\gamma)=H_1-i\gamma V_2$ has $m=k_0+k$ complex eigenvalues near 0 when $\gamma>0$ is sufficiently small. Threshold eigenvalues and resonances are unstable under perturbation and may produce both eigenvalues or quantum resonances (if the potential is dilation-analytic). Therefore, it may be interesting to see why zero eigenvalue and zero resonance of H_1 will be turned into eigenvalues of the dissipative Schrödinger operator H under weak perturbation.

Theorem 3.2. *Assume (1.1)-(1.3) with $\rho_1'>4$ and $\rho_2>4$, and (1.6).*

(a). If zero is an eigenvalue of multiplicity m, but not a resonance of H_1, then there exists $\delta,\gamma_0>0$ such that for $0<\gamma<\gamma_0$, $H(\gamma)$ has m eigenvalues in $D_-(0,\delta)$.

(b). If zero is a resonance, but not an eigenvalue of H_1, suppose in addition that the condition (1.8) is satisfied. Then for $0<\gamma<\gamma_0$, $H(\gamma)$ has one eigenvalue in $D_-(0,\delta)$.

Proof. Let H_0 denote the Friedrich's realization of $-\Delta+\frac{q(\theta)}{r^2}$ and

$$W_1(x)=V_1(x)-\frac{q(\theta)}{r^2}. \tag{3.12}$$

Then $H_1 = H_0 + W_1$. Denote $R_j(z) = (H_j - z)^{-1}$, $j = 0, 1$. The low-energy asymptotic expansion of $R_0(z)$ is calculated in [14] and is recalled in Appendix A. One has

$$R_0(z) = G_0 + z_{\nu_1} G_1 + zG_2 + O(|z|^{1+\epsilon})$$

where G_j is continuous from $H^{-1,s}$ to $H^{1,-s}$ with $s > j+1$ for $j = 0, 2$ and G_1 is a rank one operator given by

$$G_1 = c_0 \langle \eta_0, u \rangle \eta_0, \quad u \in H^{-1,s}, s > 2, \tag{3.13}$$

with η_0 defined in Introduction and

$$c_0 = -\frac{e^{-i\pi\nu_1}\Gamma(1-\nu_1)}{\nu_1 2^{2\nu_1+1}\Gamma(1+\nu_1)}, \text{ if } 0 < \nu_1 < 1; \quad c_0 = -\frac{1}{8} \text{ if } \nu_1 = 1. \tag{3.14}$$

z_{ν_1} is defined by

$$z_{\nu_1} = \begin{cases} e^{\nu_1 \ln z}, & \nu_1 \in]0, 1[\\ z \ln z, & \nu_1 = 1. \end{cases}$$

with the branch of $\ln z$ chosen such that it is holomorphic on the slit complex plane $\mathbb{C} \setminus \mathbb{R}_+$ and $\lim_{\epsilon \to 0+} \ln(\gamma + i\epsilon) = \ln \gamma$ if $\gamma > 0$. Although W_1 is not H_0-form compact, it is still H_0-form bounded by the Hardy's inequality (2.5). By the comparaison with the decomposition

$$H_1 = H_0' + W_1', \qquad \text{with } H_0' = \chi_1(-\Delta + 1)\chi_1 + \chi_2 H_0 \chi_2$$

where $\chi_1^2 + \chi_2^2 = 1$ with $\chi_2(x) = 0$ for x near 0, one can show (see [5]) that $1 + G_0 W_1$ is a Fredholm operator and its kernel, K, in $H^{1,-s}$ coincides with that of $1 + G_0' W_1'$, where

$$G_0' = \lim_{z \to 0, z \notin \mathbb{R}_+} (H_0' - z)^{-1}, \qquad \text{in } \mathcal{L}(-1, s; 1, -s).$$

Using the generalized Hardy inequality for H_0, one sees that the Hermitian form

$$K \times K \ni (\varphi, \psi) \to \langle \varphi, -W_1 \psi \rangle \in \mathbb{C}$$

is positive definite and there exists a basis $\{\phi_1, \cdots, \phi_m\}$, $m = k_0 + k$, such that

$$\langle \phi_i, -W_1 \phi_j \rangle = \delta_{ij}.$$

The assumption (1.6) implies that the multiplicity of the zero resonance is at most one. If zero is a resonance, we assume without loss that ϕ_1 is a resonant state and ϕ_j, $2 \le j \le m$, are eigenfunctions of H_1. Let $Q : H^{1,-s} \to H^{1,-s}$, $Qf = \sum_{j=1}^m \langle -W_1 \phi_j, f \rangle \phi_j$. Q is a projection from $H^{1,-s}$ onto K. Set $Q' = 1 - Q$. One can show that range of Q' is closed and is equal to the range of $1 + G_0 W_1$ in $H^{1,-s}$. Then the Fredholm theory shows that $(Q'(1 + G_0 W_1) Q')^{-1} Q'$ exists and is continuous on $H^{1,-s}$. See [5, 15].

Let $R(z, \gamma) = (H(\gamma) - z)^{-1}$, $z \notin \sigma(H(\gamma)$. One has the resolvent equations

$$R(z, \gamma) = (1 - i\gamma R_1(z) V_2)^{-1} R_1(z) = (1 + R_0(z)(W_1 - i\gamma V_2))^{-1} R_0(z).$$

The eigenvalues of $H(\gamma)$ in $D_-(0, \delta)$ are the same as the set of z for which $Hu = zu$ has a nontrivial solution u in $H^{1,-s}$ for some $s > 0$. Since $R_0(z)$ is holomorphic in $\mathbb{C}_-$, a point $z_0 \in \mathbb{C}_-$ is an eigenvalue of $H(\gamma)$ if and only if it is a pole of

$$z \to (1 + R_0(z)(W_1 - i\gamma V_2))^{-1}$$

and their multiplicities are the same. Let

$$W(z,\gamma) = 1 + R_0(z)(W_1 - i\gamma V_2).$$

Then for any $s > 1$ and s near 1,

$$W(z,\gamma) = 1 + G_0 W + O(|z|^\epsilon + \gamma)$$

in $\mathcal{L}(1,-s;1,-s)$ for $z \in D_-(0,\delta)$ and $0 < \gamma \le \gamma_0$. Since $(Q'(1+G_0W_1)Q')^{-1}Q'$ exists on $H^{1,-s}$, we deduce that for δ and γ_0 small enough,

$$E'(z,\gamma) := (Q'W(z,\gamma)Q')^{-1}Q'$$

exists on $H^{1,-s}$ and $\|E'(z,\gamma)\|_{\mathcal{L}(1,-s;1,-s)}$ is uniformly bounded for $z \in D_-(0,\delta)$ and $0 < \gamma \le \gamma_0$.

This allows us to construct for $z \in D_-(0,\delta)$ and $0 < \gamma \le \gamma_0$ the inverse for the following Grushin problem

$$\mathcal{W}(z,\gamma) = \begin{pmatrix} W(z,\gamma) & T \\ S & 0 \end{pmatrix} : H^{1,-s} \times \mathbb{C}^m \to H^{1,-s} \times \mathbb{C}^m,$$

where $s > 1$, T and S are given by

$$\begin{aligned} Tc &= \sum_{j=1}^m c_j \phi_j, \quad c = (c_1, \cdots, c_m) \in \mathbb{C}^m, \\ Sf &= (\langle -W_1\phi_1, f\rangle, \cdots, \langle -W_1\phi_m, f\rangle) \in \mathbb{C}^m, \quad f \in H^{1,-s}. \end{aligned}$$

This inverse is given by

$$\mathcal{W}(z,\gamma)^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{+-}(z) \end{pmatrix},$$

where

$$E(z) = E'(z), \tag{3.15}$$
$$E_+(z) = T - E'(z)W(z)T, \tag{3.16}$$
$$E_-(z) = S - SW(z)E'(z), \tag{3.17}$$
$$E_{+-}(z) = -SW(z)T + SW(z)E'(z)W(z)T. \tag{3.18}$$

Here to simplify notation, the indication of dependance on γ is omitted. It follows that the inverse of $W(z)$ is given by

$$W(z)^{-1} = E(z) - E_+(z)E_{+-}(z)^{-1}E_-(z) \text{ on } H^{1,-s}. \tag{3.19}$$

Since $E(z), E_\pm(z)$ and $E_{+-}(z)$ are holomorphic and uniformly bounded as operators on $H^{1,-s}$ for $z \in D_-(0,\delta)$ and $\gamma > 0$ small, z_0 is a pole of $W(z)^{-1}$ in $D_-(0,\delta)$ if and only if

$$F(z_0,\gamma) := \det E_{+-}(z_0) = 0$$

and their multiplicities are the same (cf. [2]). Since the equation $Hu = zu$ for $u \in H^{1,-s}$ and $\Im z < 0$ implies $u \in H^{1,r}$ for any $r > 0$, the poles of $z \to W(z)^{-1}$ in $\mathbb{C}_-$ as operator on $H^{1,-s}$ coincide with the eigenvalues of H.

We are now led to prove that for $0 < \gamma \leq \gamma_0$ small enough, $F(z,\gamma)$ has m zeros in $D_-(0,\delta)$. Since $\phi_j \in L^{2,-s}$ for any $s > 1$, under the condition $\rho_1' > 4$ and $\rho_2 > 4$, one can calculate the asymptotics of the matrix

$$E_{+-}(z) = (\langle -W_1\phi_j, (W(z) - W(z)E'(z)W(z))\phi_k\rangle)_{1\leq j,k\leq m}\,, \tag{3.20}$$

up to an error $O(|z|^{1+\epsilon})$. This computation is done in the case $\gamma = 0$ (see Proposition 4.4, [15]). In the case $\gamma \neq 0$, the calculation is similar. We give only the result and omit the details. One has

$$(E_{+-}(z))_{j,k} = -i\gamma v_{jk} + z_{\nu_1} a_{jk} + z(b_{jk} + r_{jk}) + O(|z|^{1+\epsilon}) \tag{3.21}$$

where

$$\begin{aligned}
v_{jk} &= \langle \phi_j, V_2\phi_k\rangle && (3.22)\\
a_{jk} &= -c_0(|c_1|^2\delta_{1j}\delta_{1k} - i\gamma\bar{c}_1 c_k'\delta_{1k}) \quad \text{with} && (3.23)\\
c_1 &= \langle W_1\eta_0, \phi_1\rangle, \quad c_j' = \langle V_2\eta_0, \phi_j\rangle, && (3.24)\\
b_{jk} &= \langle -W_1\phi_j, G_2W_1\phi_k\rangle + i\gamma\langle W_1\phi_j, G_2V_2\phi_k\rangle && (3.25)\\
r_{jk} &= \langle -W_1\phi_j, G_1(W_1 - i\gamma V_2)E'(0)G_1(W_1 - i\gamma V_2)\phi_k\rangle. && (3.26)
\end{aligned}$$

Here $\delta_{1j} = 1$ or 0 according to whether $j = 1$ or not. In the case 0 is not a resonance, $c_1 = 0$. Remark also that if ϕ_j and ϕ_k are both eigenfunctions of H_1,

$$\langle W_1\phi_j, G_2W_1\phi_k\rangle = \langle \phi_j, \phi_k\rangle \tag{3.27}$$

and if ϕ_j is an eigenfunction of H_1,

$$r_{jk} = 0, \quad k = 1, \cdots, m. \tag{3.28}$$

One can show as in Proposition 3.1 that the matrice (v_{jk}) is positive definite. In fact, it is clearly positive since $V_2 \geq 0$. If 0 is an eigenvalue of this matrix, we can take an associated eigenvector $d = (d_1, \cdots, d_m) \in \mathbb{C}^m \setminus \{0\}$. Let $\psi = \sum_j d_j\phi_j$. Then one has $V_2\psi = 0$ and $H_1\psi = 0$ which imply $H\psi = 0$. Lemma 2.1 shows that $\psi = 0$. This leads to a contradiction with the fact that $\phi_1, \cdots, \phi_m$ are linearly independent.

Consider firstly the case that zero is an eigenvalue but not a resonance of H_1. One has $a_{jk} = 0$ and

$$(E_{+-}(z))_{j,k} = -i\gamma v_{jk} + z(-\langle \phi_j, \phi_k\rangle + i\gamma\langle W_1\phi_j, G_2V_2\phi_k\rangle) + O(|z|^{1+\epsilon}) \tag{3.29}$$

Since the matrices (γv_{jk}) and $(\langle \phi_j, \phi_k\rangle)$ are positive definite, $F_0(z,\gamma) = \det(i(\gamma v_{jk}) + z(\langle \phi_j, \phi_k\rangle))$ has m zeros of the form $z = -i\gamma\sigma_j$, $\sigma_j > 0$. Let $-i\gamma\sigma$ be one of the zeros of $F_0(z,\gamma)$ with order k. For some appropriate $c > 0$ such that the distance from zeros of $z \to F_0(z,\gamma)$ to the circle $\partial D(-i\gamma\sigma, c\gamma)$ is at least $c'\gamma$ for some $c' > 0$, one has

$$|F_0(z,\gamma)| \geq C_1\gamma^m, \quad |F(z,\gamma) - F_0(z,\gamma)| \leq C_2\gamma^{m+\epsilon}$$

for $|z + i\gamma\sigma| = c\gamma$. For $\gamma > 0$ small, we can apply Rouché's theorem to $F(z,\gamma)$ to conclude that $F(z,\gamma)$ has also k zeros in the disk $D(-i\gamma\sigma, c\gamma)$. Therefore, the total number of zeros of $F(z,\gamma)$ in $D_-(0,\delta)$ are at least m. If z_0 is a zero of $F(z,\gamma)$, the asymptotic expansion of $F(z,\gamma)$ in z shows that $\varsigma = \lim_{\gamma\to 0} z_0/\gamma$ exists and $\gamma\varsigma$ is a zero of $F_0(z,\gamma)$. This allows to conclude that $F(z,\gamma)$ has m zeros in $D_-(0,\delta)$ for $\delta > 0$ and

$0 < \gamma < \gamma_0$ small enough.

Now assume that zero is a resonance, but not an eigenvalue of H_1. $E_{+-}(z)$ is a scalar function, holomorphic in $z \in D_-(0,\delta)$ and

$$E_{+-}(z) = -i\gamma v_{11} - c_0 z_{\nu_1}(|c_1|^2 - i\gamma \bar{c}_1 c_1') + O(|z|), \tag{3.30}$$

where $v_{11} > 0$. Assume (1.8). The root of the equation

$$i\gamma v_{11} + c_0 z_{\nu_1}(|c_1|^2 - i\gamma \bar{c}_1 c_1') = 0. \tag{3.31}$$

can be explicitly calculated. Set

$$r = \frac{\gamma v_{11}}{|c_0|\;|(|c_1|^2 - i\gamma \bar{c}_1 c_1')|}.$$

Since $\bar{c}_1 c_1' < 0$ by the assumption (1.8), the argument, φ, of $-\gamma \bar{c}_1 c_1' + i|c_1|^2$ verifies

$$0 < \varphi < \frac{\pi}{2}, \quad \lim_{\gamma \to 0_+} \varphi = \frac{\pi}{2}. \tag{3.32}$$

With the above definitions of r and φ, (3.31) can be rewritten as

$$z_{\nu_1} = -\frac{i\gamma v_{11}}{c_0(|c_1|^2 - i\gamma \bar{c}_1 c_1')} = re^{i(\varphi + \pi - \arg c_0)}. \tag{3.33}$$

For $\nu_1 \in]0,1[$, $z_{\nu_1} = z^{\nu_1}$ and $c_0 = -\frac{e^{-i\pi\nu_1}\Gamma(1-\nu_1)}{\nu_1 2^{2\nu_1+1}\Gamma(1+\nu_1)}$. (3.33) becomes

$$z^{\nu_1} = re^{i(\pi\nu_1 + \varphi)}. \tag{3.34}$$

When $\nu_1 \in [1/2, 1[$, (3.34) has a unique solution $z' = \rho e^{i\theta}$ in $D_-(0,\delta)$ given by

$$\rho = r^{1/\nu_1} \quad \text{and} \quad \theta = \pi + \nu_1^{-1}\varphi. \tag{3.35}$$

The above calculation gives the location of this root:

$$z' = d_1 \gamma^{\frac{1}{\nu_1}} e^{i(\pi + \frac{\pi}{2\nu_1} - \gamma d_2)}(1 + o(1)) \tag{3.36}$$

for some $d_1, d_2 > 0$.

When $\nu_1 = 1$, $z_{\nu_1} = z \ln z$ and $c_0 = -\frac{1}{8}$. $z' = \rho e^{i\theta}$ is a solution of (3.33) if

$$\rho e^{i\theta}(\ln \rho + i\theta) = re^{i\varphi} \tag{3.37}$$

Remark that $r = O(\gamma)$ as $\gamma \to 0$ and $\arg(\ln \rho + i\theta) \to \pi_-$ as $\rho \to 0_+$. Set $\ln z' = \tau e^{i(-\pi-\sigma)}$ with $\tau = |\ln z'|$. For $\gamma > 0$ small enough, one can check that the system

$$\begin{cases} \tau e^{-\tau \cos\sigma} & = & r, \\ -\sigma + \tau \sin\sigma & = & \varphi + \pi \end{cases}$$

has a unique solution (τ, σ) such that $\tau \to \infty$ and $\sigma \to 0_+$ as $\gamma \to 0$. This shows that (3.37) has a unique solution $z' = \rho e^{i\theta}$ in $D_-(0,\delta)$ given by

$$\rho = |e^{\tau e^{i(-\pi-\sigma)}}| = e^{-\tau\cos\sigma} \quad \text{and} \quad \theta = \pi + \varphi - \sigma.$$

Using Rouché's theorem, one can show as before that $E_{+-}(z,\gamma)$ has just one zero in $D_-(0,\delta)$ which is located inside the small disk $D(z_0, g(\gamma))$, where $g(\gamma) = a\gamma^{1/\nu_1}$ if $\nu_1 \in]0,1[$ and $g(\gamma) = a\gamma\langle \ln\gamma \rangle^{-1}$ if $\nu_1 = 1$, $a > 0$ being an appropriate constant. This proves that $H(\gamma)$ has exactly one eigenvalue in $D_-(0,\delta)$ and it is simple. $\square$

Theorem 1.2 follows from Theorem 3.2 and the argument used Proposition 3.1 outside a small neighborhood of zero.

The case that zero is both an eigenvalue and a resonance can in principle be analyzed in a similar way, using the asymptotic expansion of $E_{+-}(z,\gamma)$ given above. But the evaluation of zeros of the determinant raises some technical difficulties when m is arbitrary. Without the assumption (1.6), zero resonance of H_1 may appear with arbitrary multiplicity. The matrix $E_{+-}(z,\gamma)$ can be still calculated, but its analysis is more complicated. We do not go further here.

Remark 3.3. *An example for which zero is not an eigenvalue, but a resonance of $H_1 = -\Delta + V_1$ can be constructed as follows. Let $n = 3$ or 4 and let v_1 be a real-valued function satisfying (1.1) with $\rho_1 > 4$ and $v_1 \leq 0$, $v_1 \neq 0$. Let $H_1(\beta) = -\Delta + \beta v_1$. Then one can show that there exists a critical constant $\beta_0 > 0$ such that $H_1(\beta_0) \geq 0$ and $H_1(\beta)$ has at least one negative eigenvalue for any $\beta > \beta_0$. Then zero is a resonance but not an eigenvalue of $H_1(\beta_0)$. A resonant state of $H_1 = -\Delta + \beta_0 v_1$ can be constructed as weak limit of the fundamental state of $H_1(\beta)$ as $\beta \to \beta_{0+}$ and it does not change sign. The condition (1.8) is then satisfied for $V_1 = \beta_0 v_1$ and for any $V_2 \in C_0^\infty(\mathbb{R}^n)$ with $V_2 \geq 0$ and $V_2 \neq 0$. In this example, the number of eigenvalues of H_1 is zero, while that of $H_1 - i\gamma V_2$ is one for any $\gamma > 0$ small enough.*

Appendix A. Low-energy resolvent expansion on conical manifolds

In this appendix, we recall in a concise way the result of [14] on low-energy resolvent expansion of the model operator. Consider the operator

$$P_0 = -\Delta_g + \frac{q(\theta)}{r^2} \tag{A.1}$$

on a conical manifold $M = \mathbb{R}_+ \times \mathbf{\Sigma}$ equipped with a Riemannian metric g, where $\mathbf{\Sigma}$ is an $(n-1)$-dimensional compact manifold, $n \geq 2$. Here $(r,\theta) \in \mathbb{R}_+ \times \mathbf{\Sigma}$, $q(\theta)$ is a real continuous function and the metric g is of the form

$$g = dr^2 + r^2 h$$

with h a Riemannian metric on $\mathbf{\Sigma}$ independent of r. If $\mathbf{\Sigma}$ is of boundary, the Dirichlet condition is used for P_0. We still denote by P_0 its Friedrich's realization with the core $C_0^\infty(\mathbb{R}^n \setminus \{0\})$. Let Δ_h denote Laplace-Beltrami operator on $\mathbf{\Sigma}$. Assume

$$-\Delta_h + q(\theta) \geq -\frac{(n-2)^2}{4}, \qquad \text{on } L^2(\mathbf{\Sigma}). \tag{A.2}$$

Put

$$\sigma_\infty = \left\{ \nu;\ \nu = \sqrt{\lambda + \frac{(n-2)^2}{4}}, \lambda \in \sigma(-\Delta_h + q) \right\}. \tag{A.3}$$

Denote

$$\sigma_k = \sigma_\infty \cap [0,k], \qquad k \in \mathbb{N}.$$

Let π_ν denote the orthogonal projection in $L^2(M)$ onto the subspace spanned by the eigenfunction of $-\Delta_h + q$ associated with the eigenvalue $\lambda_\nu = \nu^2 - \frac{(n-2)^2}{4}$. Define for $\nu \in \sigma_\infty$

$$z_\nu = \begin{cases} z^{\nu'}, & \text{if } \nu \notin \mathbb{N} \\ z \ln z, & \text{if } \nu \in \mathbb{N}^*, \end{cases}$$

where $\nu' = \nu - [\nu]$. Let $\sigma_N = \sigma_\infty \cap [0, N]$. For $\nu > 0$, let $[\nu]_-$ be the largest integer strictly less than ν. When $\nu = 0$, set $[\nu]_- = 0$. Define δ_ν by $\delta_\nu = 1$, if $\nu \in \mathbb{N} \cap \sigma_\infty$; 0, otherwise.

Theorem A.1. *Let $R_0(z) = (P_0 - z)^{-1}$ for $z \notin \mathbb{R}_+$. The following asymptotic expansion holds for z near 0 with $z \notin \mathbb{R}_+$.*

$$R_0(z) = \delta_0 \ln z \, G_{0,0} + \sum_{j=0}^{N} z^j F_j + \sum_{\nu \in \sigma_N} z_\nu \sum_{j=[\nu]_-}^{N-1} z^j G_{\nu, j+\delta_\nu} \pi_\nu + O(|z|^{N+\epsilon}), \tag{A.4}$$

in $\mathcal{L}(-1, s; 1, -s)$, $s > 2N + 1$. Each term in the above expansion can be explicitly calculated. In particular, $F_j \in \mathcal{L}(-1, s; 1, -s)$, $s > 2j + 1$ and $G_{\nu,j}$, $j \geq [\nu]_-$, is of finite rank with its Schwartz kernel on $L^2(\mathbb{R}_+; r^{n-1}dr)$ given by

$$G_{\nu,j}(r, \tau) = \frac{(-1)^{j+1-[\nu]} e^{-i\nu'\pi} \Gamma(1 - \nu') \ (r\tau)^{-\frac{n-2}{2} + \nu' + j}}{2^{2\nu+1} \pi^{\frac{1}{2}} (j - [\nu])! \ \Gamma(\frac{1}{2} + \nu) \nu'(\nu' + 1) \cdots (\nu' + j)} P_{\nu, j-[\nu]}(\rho) \tag{A.5}$$

for $\nu \notin \mathbb{N}$, $j \geq [\nu]$ and $\nu' = \nu - [\nu]$; and

$$G_{\nu,j}(r, \tau) = \frac{(-1)^{j+l+1}}{\pi^{\frac{1}{2}} 2^{2l+1} j! \ (j - l)! \ \Gamma(l + \frac{1}{2})} (r\tau)^{-\frac{n-2}{2} + j} P_{l, j-l}(\rho) \tag{A.6}$$

for $\nu = l \in \mathbb{N}$ and $j \geq l$. $P_{\nu,k}(\rho)$ is a polynomial of degree k in ρ:

$$P_{\nu,k}(\rho) = \int_{-1}^{1} (\rho + \frac{\theta}{2})^k (1 - \theta^2)^{\nu - 1/2} d\theta, \quad \rho = \frac{r^2 + \tau^2}{4r\tau}. \tag{A.7}$$

Note that

$$P_{\nu,0}(\rho) = \frac{\Gamma(1/2)\Gamma(1/2 + \nu)}{\Gamma(1 + \nu)}, \quad P_{\nu,1}(\rho) = \frac{\Gamma(1/2)\Gamma(1/2 + \nu)}{\Gamma(1 + \nu)} \rho. \tag{A.8}$$

One has for $\nu \in [0, 1]$

$$G_{\nu,[\nu]}(r, \tau) = \gamma_\nu (r\tau)^{-\frac{n-2}{2} + \nu}, \tag{A.9}$$

with

$$\gamma_\nu = -\frac{e^{-i\pi\nu} \Gamma(1 - \nu)}{\nu 2^{2\nu+1} \Gamma(1 + \nu)} \quad \text{for } \nu \in]0, 1[, \quad \gamma_0 = -\frac{1}{2} \quad \text{and} \quad \gamma_1 = -\frac{1}{8}. \tag{A.10}$$

The expansion of $R_0(z)$ is obtained by decomposing $R_0(z)$ into

$$R_0(z) = \sum_{\nu \in \sigma_\infty} (Q_\nu - z)^{-1} \pi_\nu$$

with

$$Q_\nu = -\frac{d^2}{dr^2} - \frac{n-1}{r} \frac{d}{dr} + \frac{\nu^2 - \frac{(n-2)^2}{4}}{r^2}, \quad \text{on } L^2(\mathbb{R}_+; r^{n-1}dr).$$

Since the Schwartz kernel of e^{-itQ_ν} is given by

$$\frac{1}{2it}(r\tau)^{-\frac{n-2}{2}}e^{-\frac{r^2+\tau^2}{4it}-i\frac{\pi\nu}{2}}J_\nu(\frac{r\tau}{2t}), \quad t\in\mathbb{R}, \tag{A.11}$$

where $J_\nu(\cdot)$ is the Bessel function of the first kind of order ν (see [13]) and

$$(Q_\nu - z)^{-1} = i\int_0^\infty e^{-it(Q_\nu - z)}\,dt$$

for $\Im z > 0$, the Schwartz kernel of $(Q_\nu - z)^{-1}$ is

$$K_\nu(r,\tau;z) \;=\; (r\tau)^{-\frac{n-2}{2}}\int_0^\infty e^{-\frac{r^2+\tau^2}{4it}+izt-i\frac{\pi\nu}{2}}J_\nu(\frac{r\tau}{2t})\,\frac{dt}{2t}. \tag{A.12}$$

for $\Im z > 0$. The formula (2.6) in [14] for this kernel contains a wrong sign. The coefficients in (A.5) and (A.6) are obtained from the constants given in Section 2 and Appendix A of [14], in taking into account some sign corrections. Note that under the assumption (1.8), the continuity of F_j and the remainder estimate can be improved. See Remark 2.4 in [14].

Acknowledgement The author thanks the anonymous referee for his or her kind remark and suggestion on the proof of Lemma 2.1.

References

[1] S. Agmon, Spectral properties of Schrödinger operators and scattering theory. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) (1975), no. 2, 151-218.

[2] B. Helffer, J. Sjöstrand, Résonances en limite semi-classique, Bull. Soc. Math. France, Mémoire No.24/25, 114(1986).

[3] A. Ionescu and D. Jerison, On the absence of positive eigenvalues of Schrödinger operators with rough potentials, Geom. and Funct. Anal., 13(2003), 1029-181.

[4] A. Jensen and T. Kato, Spectral properties of Schrödinger operators and time decay of wave functions, Duke Math. J., 46(1979), 583-611.

[5] X.Y. Jia, Some threshold spectral problems for Schrödinger operators, PhD thesis, Nantes, 2009.

[6] T. Kako, K. Yajima, Spectral and scattering theory for a class of non-selfadjoint operators. Sci. Papers College Gen. Ed. Univ. Tokyo 26 (1976), no. 2, 73-89

[7] T. Kato, Wave operators and similarity for some non-selfadjoint operators, Math. Annalen, 162(1966), 258-279.

[8] A. Laptev, O. Safronov, Eigenvalues estimates for Schrödinger operators with complex potentials, Comm. Math. Phys. 292 (2009), no. 1, 29-54.

[9] G. Rozenblum, M. Solomyak, Counting Schrödinger boundstates: semiclassics and beyond. Sobolev spaces in mathematics. II, 329-353, Int. Math. Ser. (N. Y.), 9, Springer, New York, 2009.

[10] J. Royer, Limiting absorption principle for the dissipative Helmholtz equation, Commun. in PDE, to appear.

[11] Y. Saito, The principle of limiting absorption for the nonselfadjoint Schrödinger operator in $R^N (N\neq 2)$. Publ. Res. Inst. Math. Sci. 9 (1973/74), 397-428.

[12] J. T. Schwartz, Some non-selfadjoint operators, Comm. Pure and Appl. Math., 13(1960), 609-639.

[13] M. Taylor, *Partial differential equations II*, New York, Springer 1997.

[14] X. P. Wang, Threshold energy resonance in geometric scattering, Proceedings of Symposium "Scattering and Spectral Theory", August 2003, Recife, Brazil. Matemática Contemporânea, 26 (2004), 135-164.

[15] X.P. Wang: Asymptotic expansion in time of the Schrödinger group on conical manifolds, Ann, Inst. Fourier (Grenoble), 56(2006), 1903-1945

Département de Mathématiques, UMR 6629 CNRS, Université de Nantes, 44322 Nantes Cedex 3 France, E-mail: xue-ping.wang@univ-nantes.fr